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Energy flow in a bound electromagnetic field: resolution of apparent paradoxes

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Abstract
In this paper, we present a resolution of apparent paradoxes formulated in (Kholmetskii A L 2006 Apparent paradoxes in classical electrodynamics: the energy–momentum conservation law for a bound electromagnetic field *Eur. J. Phys.* **27** 825–38; Kholmetskii A L and Yarman T 2008 Apparent paradoxes in classical electrodynamics: a fluid medium in an electromagnetic field *Eur. J. Phys.* **29** 1127) and dealing with the energy flux in a bound electromagnetic field.

1. Introduction

In this contribution, we continue to resolve the apparent paradoxes of classical electrodynamics, described in our previous papers [1–3]. Now we select the problems, where the electromagnetic energy flux and electromagnetic momentum were involved with the formulation of paradoxes. In section 2 we analyse the problem, which opened the entire series of our papers on apparent paradoxes in classical electromagnetism [1]: the energy flux in an electromagnetic (EM) field of an isolated charged particle, moving at a constant velocity in an inertial frame of observation. In section 3 we resolve the paradoxes formulated for a moving parallel-plate charged capacitor, where the electric attraction of the plates is balanced by the pressure of gas convicted between these. In section 4 we analyse some general problems concerning the EM energy–momentum tensor and section 5 deals with concluding remarks.

2. Electromagnetic energy flow of a moving isolated charged particle

This paradox has been presented in [1], and its essence is as follows. Let an isolated charged particle q move at a constant velocity $\boldsymbol{v}$ in an inertial frame of observation. The particle produces a velocity-dependent (bound) electric field $\boldsymbol{E}$ as well as a bound magnetic field

$$\boldsymbol{B} = \boldsymbol{v} \times \boldsymbol{E}/c. \tag{1}$$

The EM energy density u and energy flux density (Poynting vector) $\boldsymbol{S}$ at each spatial point are determined by the familiar expressions

$$u = \frac{E^2 + B^2}{8\pi} \tag{2}$$

$$\boldsymbol{S} = \frac{c(\boldsymbol{E} \times \boldsymbol{B})}{4\pi}, \tag{3}$$

respectively. Further, we marked out a closed spatial volume V *outside* the particle. The law of energy conservation for this volume is expressed as

$$\int_V \frac{\partial u}{\partial t} \mathrm{d}V = -\int_\xi \boldsymbol{S}\, d\xi,$$

where ξ is its boundary area. The latter equation yields

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} = 0, \tag{4}$$

which, according to the known theorem of vector analysis [4], represents a sufficient condition that u and $\boldsymbol{S}$ compose a 4-vector. However, the isolated charged particle moving at a constant velocity produces the bound EM field only, where the energy density u and energy flux density $\boldsymbol{S}$ transform not as 4-vectors, but rather as the time-like components of a symmetric tensor [1, 4]. Thus, we get a paradoxical situation.

This paradox is closely related to the familiar '4/3 puzzling' [5] and stems from the fact that the bound EM field itself does not compose an isolated system. Therefore, the four-divergence of the EM energy–momentum tensor for such a field does not vanish. We can check by direct calculations that equation (3) is not fulfilled for the bound EM field indeed, substituting (1)–(3) into (4). Hence

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{E^2 + B^2}{8\pi}\right) = \frac{1}{4\pi}\left(\boldsymbol{E}\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{B}\frac{\partial \boldsymbol{B}}{\partial t}\right) = -\frac{1}{4\pi}(\boldsymbol{E}(\boldsymbol{v} \cdot \nabla)\boldsymbol{E} + \boldsymbol{B}(\boldsymbol{v} \cdot \nabla)\boldsymbol{B}). \tag{5}$$

In this equation we assume a dependence of the EM field on the present time coordinates. Besides, we have taken into account that both fields $\boldsymbol{E}$ and $\boldsymbol{B}$ implicitly depend on time, so that $\frac{\partial}{\partial t} = -(\boldsymbol{v} \cdot \nabla)$. Further, we direct the x-axis along the vector $\boldsymbol{v}$ and derive

$$\frac{\partial u}{\partial t} = -\frac{v}{4\pi}\left(E_x\frac{\partial E_x}{\partial x} + E_y\frac{\partial E_y}{\partial x} + E_z\frac{\partial E_z}{\partial x} + B_x\frac{\partial B_x}{\partial x} + B_y\frac{\partial B_y}{\partial x} + B_z\frac{\partial B_z}{\partial x}\right). \tag{6}$$

Using equation (1), we express the components of $\boldsymbol{B}$-field via the components of $\boldsymbol{E}$-field:

$$B_x = 0, \qquad B_y = -\frac{v}{c}E_z, \qquad B_z = \frac{v}{c}E_y. \tag{7}$$

Inserting equations (7) into (6), we obtain

$$\frac{\partial u}{\partial t} = -\frac{v}{4\pi}\left(E_x\frac{\partial E_x}{\partial x} + E_y\frac{\partial E_y}{\partial x}\left(1 + \frac{v^2}{c^2}\right) + E_z\frac{\partial E_z}{\partial x}\left(1 + \frac{v^2}{c^2}\right)\right). \tag{8}$$

Combining equations (1) and (3), and applying the vector identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \bullet \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \bullet \boldsymbol{b})$, we get an expression for $\boldsymbol{S}$ to be convenient for further manipulations:

$$\boldsymbol{S} = \frac{(\boldsymbol{E} \times (\boldsymbol{v} \times \boldsymbol{E}))}{4\pi} = \frac{1}{4\pi}(\boldsymbol{v}E^2) - \frac{1}{4\pi}\boldsymbol{E}(\boldsymbol{v} \bullet \boldsymbol{E}).$$

Hence

$$\nabla \cdot \boldsymbol{S} = \frac{1}{4\pi}[(\boldsymbol{v} \cdot \nabla E^2) - \boldsymbol{E} \cdot \nabla(\boldsymbol{v} \bullet \boldsymbol{E})]. \tag{9}$$

In the latter equation, we have taken into account that inside the volume V, $\nabla \cdot \boldsymbol{E} = 0$. Writing equation (9) in components, we arrive at

$$\nabla \cdot \boldsymbol{S} = \frac{v}{4\pi}\left[E_x\frac{\partial E_x}{\partial x} + 2E_y\frac{\partial E_y}{\partial x} + 2E_z\frac{\partial E_z}{\partial x} - E_y\frac{\partial E_x}{\partial y} - E_z\frac{\partial E_x}{\partial z}\right]. \tag{10}$$

We see that, in general, the sum of equations (6) and (10) does not vanish, and the equality (4) is not fulfilled indeed. Thus the paradox persists.

For its resolution we first recall that in the Lienard–Wiechert expression for an electric field of a point-like charge q, the velocity-dependent term can be expressed via the present time coordinates, when $\boldsymbol{v}$ is a constant. Physically this result is quite understandable, because for any known law of motion (in particular, for $\boldsymbol{v} = \text{const.}$), the retarded and present quantities can be unambiguously related to each other. In such present time coordinates (assumed also in equations (5)–(10)) the electric field is described by the Heaviside formula [6]

$$\boldsymbol{E} = \frac{q(1 - v^2/c^2)\boldsymbol{r}}{\left(1 - \frac{v^2}{c^2}\sin^2\vartheta\right)^{3/2} \boldsymbol{r}^3}, \tag{11}$$

with ϑ being the polar angle. This equation shows that the electric field $\boldsymbol{E}$ is directed along the radius vector joining the present position of a charge q with the point of observation, and it *rigidly* propagates with a source charge. The magnetic field is determined by equation (1) and also *rigidly* propagates with the charge. At the same time, we have to be careful with the physical interpretation of this result. If we take two near points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, so that the vector $\boldsymbol{r}_2 - \boldsymbol{r}_1$ is parallel to the direction of motion of the charge, then equation (11) tells us that the electric field $\boldsymbol{E}(\boldsymbol{r}_1, t)$ measured in the point $\boldsymbol{r}_1$ at the moment t, is equal to $\boldsymbol{E}(\boldsymbol{r}_2, t + \Delta t)$, if $\Delta t = |\boldsymbol{r}_2 - \boldsymbol{r}_1|/v$. However, this result does not signify as yet that the electric field itself propagates at the velocity $\boldsymbol{v}$; it is well known that the EM field 'knows' only two velocities: the light velocity c and zero velocity (for example, the latter case is relevant for a bound EM field of a resting charge). Thus, it should be recognized that the equality $\boldsymbol{E}(\boldsymbol{r}_1, t) = \boldsymbol{E}\left(\boldsymbol{r}_2, t + \frac{|r_2 - r_1|}{v}\right)$ appears as a result of propagation of the EM field at the light velocity c from the retarded positions of source charge to the corresponding points of observation. However, this interpretation is not relevant for the energy flow: its direction is guided by the Poynting vector $\boldsymbol{S}$, which is not collinear, but rather orthogonal, to the line joining the point of observation and present position of charge. The question is as follows: how do we observe in these conditions the velocity of EM energy flow different from c? The answer to this question may be simple, if we use an analogy from the fluid mechanics. It is easy to imagine that various marked parts of some fluid medium have different instantaneous velocities, but the medium itself moves in a frame of observation at some average *effective velocity* $\boldsymbol{v}$, which might coincide with none of the velocities of its parts. The same could be true for the bound EM field, where its effective velocity v is composed from two values: 0 and c.[3] Thus, in order to explain equation (11) (a rigid propagation of the bound EM field synchronically with its source particle) in terms of the energy flow, we have to admit that this field, like a fluid medium, propagates by means of energy 'streams' with the velocity c, which involve only some fractions of the total field, while other fractions of the field remain at instantaneous rest (to our knowledge, this interpretation was first proposed in [7]). Propagation of such 'streams' is described by the Poynting vector (3), but the field as a whole moves at the *effective velocity* of the source charge $\boldsymbol{v}$. Going from the effective velocity of the EM field to its *effective energy flux*, we now intend to derive a mathematical expression for the latter, when

[3] This approach, of course, is not applicable to an EM radiation, where a single admissible velocity c exists.

the isolated charged particle is considered. For this purpose we need to modify the Poynting theorem,

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} + \boldsymbol{j} \cdot \boldsymbol{E} = 0, \tag{12}$$

to a new equation, which should contain a partial time derivative of the energy density and a divergence of some vector $\boldsymbol{S}_{ef}$. If so, we get a right to identify $\boldsymbol{S}_{ef}$ with the *effective energy flux density* of the bound EM field.

First we write

$$\nabla \cdot \boldsymbol{S} = \frac{c}{4\pi} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \frac{c}{4\pi} [\boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B})], \tag{13}$$

where the vector identity $\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\nabla \times \boldsymbol{a}) - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b})$ has been used. Applying the Maxwell equations, $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/c\partial t$ and $\nabla \cdot \boldsymbol{E} = 4\pi\rho$, and using the equality,

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{E}) = \boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{E}) - (\boldsymbol{v} \cdot \nabla)\boldsymbol{E} = 4\pi\rho\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{E} = 4\pi\boldsymbol{j} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{E},$$

we further obtain

$$\begin{aligned} \nabla \cdot \boldsymbol{S} &= -\frac{1}{4\pi}\left[\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E} \cdot (\nabla \times (\boldsymbol{v} \times \boldsymbol{E}))\right] \\ &= -\frac{\boldsymbol{B}}{4\pi c} \cdot \left(\boldsymbol{v} \times \frac{\partial \boldsymbol{E}}{\partial t}\right) - \boldsymbol{E} \cdot \boldsymbol{j} + \frac{1}{4\pi} \boldsymbol{E} \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{E}]. \end{aligned} \tag{14}$$

Using the next Maxwell equation $\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}$, we compute the first term in the rhs of equation (14)

$$\boldsymbol{B} \cdot (\boldsymbol{v} \times \partial \boldsymbol{E}/\partial t) = \boldsymbol{B} \cdot [\boldsymbol{v} \times (c(\nabla \times \boldsymbol{B}) - 4\pi\boldsymbol{j})] = -c\boldsymbol{B} \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{B}]. \tag{15}$$

Now we have taken into account that for the isolated charged particle, the vectors $\boldsymbol{v}$ and $\boldsymbol{B}$ are orthogonal to each other, so that $\boldsymbol{j} \cdot \boldsymbol{B} = \rho\boldsymbol{v} \cdot \boldsymbol{B} = 0$. Combining equations (14) and (15), one gets

$$\nabla \cdot \boldsymbol{S} = \frac{1}{4\pi} \boldsymbol{E} \cdot [(\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{E}] + \frac{1}{4\pi} \boldsymbol{B} \cdot [(\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{B}] - \boldsymbol{E} \cdot \boldsymbol{j}. \tag{16}$$

In its turn, one can see that

$$\frac{1}{4\pi} \boldsymbol{E} \cdot [(\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{E}] + \frac{1}{4\pi} \boldsymbol{B} \cdot [(\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{B}] = \nabla \cdot \left[\boldsymbol{v}\left(\frac{E^2 + B^2}{8\pi}\right)\right] = \nabla \cdot (\boldsymbol{v}u).$$

Then

$$\nabla \cdot \boldsymbol{S} = \nabla \cdot (\boldsymbol{v}u) - \boldsymbol{E} \cdot \boldsymbol{j}. \tag{17}$$

Substituting the term $\nabla \cdot \boldsymbol{S}$ from (17) into equation (12), we arrive at the equality

$$\frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{v}u) = 0. \tag{18}$$

Thus, we are successful to transform the energy balance equation (12) to the form (18), which contains the partial time derivative of u and divergence of the vector $\boldsymbol{v}u$. Hence, we can identify the vector

$$\boldsymbol{S}_{ef} = \boldsymbol{v}u \tag{19}$$

with the effective energy flux density in the bound EM field, and note that equation (18) exactly coincides in its form with Umov's theorem on the energy flow in fluid media [8]. It is important that equation (18) represents a mathematical equivalent to equation (12) and hence, it is also yielded by the Maxwell equations. The 4-vector transformation for u and $\boldsymbol{S}_{ef}$ is obvious, and the paradox considered has been resolved.

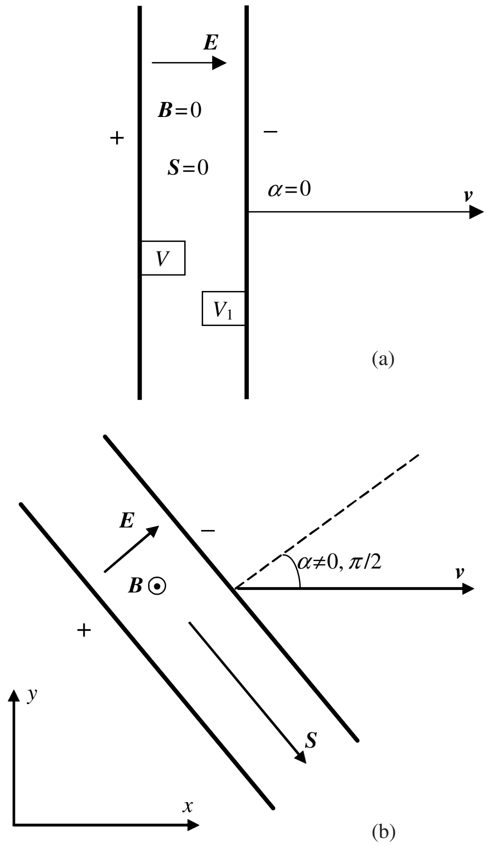

Figure 1. A parallel-plate charged capacitor, moving at a constant velocity $\boldsymbol{v}$ along the x-axis: (a) the vectors $\boldsymbol{v}$ and $\boldsymbol{E}$ constitute the angle $\alpha = 0$; (b) the vectors $\boldsymbol{v}$ and $\boldsymbol{E}$ constitute the angle $0 < \alpha < \pi/2$ in the plane xy. For both cases the direction of magnetic field $\boldsymbol{B}$ and Poynting vector $\boldsymbol{S}$ is shown (the lower diagram corresponds to the non-relativistic limit $v \ll c$).

We mention that some authors, following [9], introduce such a parameter, such as the energy flow velocity $\boldsymbol{V}$, defined as a ratio of the energy flux density and energy density:

$$\boldsymbol{V} = \boldsymbol{S}/u. \tag{20}$$

When the *effective* energy flux density (19) is substituted into equation (20), the velocity V coincides with v. However, when the Poynting vector (3) is used in equation (20), we should be rather careful in physical interpretation of V; a *real* velocity of bound EM field acquires only two values: 0 and c.

3. Parallel-plate charged capacitor, moving at a constant velocity

In [3], we have formulated two apparent paradoxes on a moving parallel-plate charged capacitor C with a different orientation of its inner electric field $\boldsymbol{E}$ and velocity $\boldsymbol{v}$ (see figure 1). The electric attraction of the plates of the capacitor is balanced by the pressure of gas convicted between these, and we adopt that the distance between the plates is much smaller than their linear sizes, and any boundary effects are negligible. The idea to analyse an implementation of the energy–momentum conservation law for various angles α between $\boldsymbol{E}$ and $\boldsymbol{v}$ was suggested by an essential difference in transformation of $T^{\mu v}_{\mathrm{EM}}$ and $(T_M)^{\mu v}$ (matter tensor of gas, μ, $v = 0, \ldots, 3$) with respect to spatial rotations, demonstrated in [3]. Namely, the tensor $(T_M)^{\mu v}$ remains unchanged, while $T^{\mu v}_{\mathrm{EM}}$ is modified by spatial rotations. As a reflection of this fact, we observed different values and directions of the Poynting vector (3) inside the capacitor for various α, and formulated two apparent paradoxes.

(1) (Figure 1(a), $\alpha = 0$): in the inner volume of the capacitor $\boldsymbol{S} = 0$, but the EM field convicted between the plates certainly changes its spatial location, while the capacitor moves.

(2) (Figure 1(b), $0 < \alpha < \pi/2$): in the non-relativistic limit, the Poynting vector $\boldsymbol{S}$ in the inner volume of C is collinear to its plates and thus, the momentum of the EM field, concentrated between the plates, has the non-vanished component onto the y-axis. However, according to the Lorentz transformation between the rest frame of C and a laboratory frame, the y-component of the total momentum of the configuration should be equal to zero.

Below we resolve both paradoxes.

3.1. The case where vectors $\boldsymbol{E}$ and $\boldsymbol{v}$ are collinear to each other (figure 1(a))

The paradox disappears, if we take a closer look at the Poynting theorem (12) and realize that the electromagnetic energy density u can be changed not only due to the energy flow, but also due to the work done $\boldsymbol{j} \cdot \boldsymbol{E}$.

Consider a small cylindrical volume V, as is shown in figure 1(a). The left-hand side of this volume is attached to the left capacitor's plate and cuts the circular area ξ on this plate, while the right-hand side of V is fixed in space and lies inside C. Thus, while the plate is moving, the volume V decreases with time. Now let us apply the integral Poynting theorem to the volume V:

$$\frac{\partial}{\partial t}\int_V u\,\mathrm{d}V + \int_V \nabla \cdot \boldsymbol{S}\,\mathrm{d}V + \int_V \boldsymbol{j} \cdot \boldsymbol{E}_-\,\mathrm{d}V = 0, \tag{21}$$

where we have taken into account that the work on the charges of the left capacitor's plate is done due to the electric field $\boldsymbol{E}_-$ of the right plate alone

$$\boldsymbol{E}_- = 2\pi\sigma, \tag{22}$$

with σ being the surface charge density of the plates. When the capacitor moves along the x-axis at the constant velocity v,

$$\frac{\partial}{\partial t}\int_V u\,\mathrm{d}V = u\frac{\mathrm{d}V}{\mathrm{d}t} = -u\xi v. \tag{23}$$

Taking also into account that

$$\int_V \boldsymbol{j} \cdot \boldsymbol{E}_-\,\mathrm{d}V = E_- v\int_V \rho\,\mathrm{d}V = E_- v\sigma\xi, \tag{24}$$

and applying the equality $\boldsymbol{S} = 0$ inside and outside the capacitor (the boundary effects, as before, are neglected), we further obtain with equations (21)–(24):

$$u = 2\pi\sigma^2. \tag{25}$$

Insofar as the surface charge density is related to the total electric field of C by the expression $\sigma = \frac{E}{4\pi}$, then the equality (25) yields

$$u = \frac{E^2}{8\pi}, \tag{26}$$

as it should be for the EM energy density inside the capacitor in figure 1(a).

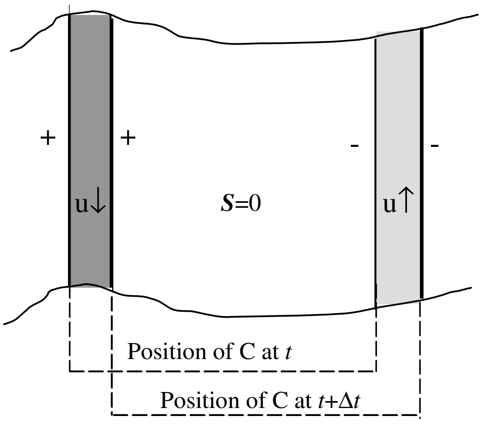

Figure 2. Position of a designated fragment of the capacitor at the time moments t and $t + \Delta t$, correspondingly.

Thus we have shown that the Poynting theorem (21) is exactly fulfilled for the marked volume V. Analogously, one can demonstrate that this theorem is also implemented for a similar volume V_1 attached to the right plate of C. Physically, it means that the electric field convicted between the capacitor's plates indeed changes its spatial location, while the capacitor is moving, but not due to the electromagnetic energy flux ($\boldsymbol{S} = 0$). Rather a motion of EM field occurs due to the work done on the charges of the left and right plates of the capacitor.

Thus the paradox considered has been resolved, but there are some interesting points disclosed in its resolution.

First we note that equation (21) differs, in general, from the standard notation of the Poynting theorem in an integral form

$$\int_V \frac{\partial u}{\partial t}\,\mathrm{d}V + \int_V \nabla\cdot\boldsymbol{S}\,\mathrm{d}V + \int_V \boldsymbol{j}\cdot\boldsymbol{E}_-\,\mathrm{d}V = 0. \tag{27}$$

This equation usually implies that the volume of integration V is fixed, so that the operator of the partial time derivative in equation (21) acts only on u and can be placed inside the first integral. In general, we are free to take an appropriate fixed volume V in figure 1(a) and to apply the Poynting theorem in the form (27) instead of (21). However, in this way we face a problem as follows. While the capacitor moves along the x-axis, the energy density u implicitly depends on time and $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}v$. Classically, the EM energy density u as a function of distance is discontinuous on the plane of the capacitor's plates: it is equal to zero outside the capacitor and acquires a finite value in its inner volume, so that the partial spatial derivative $\partial u/\partial x$ on the capacitor's plates becomes uncertain. Thus, the first integral on the lhs of equation (27) cannot be evaluated, whereas we have no problem to compute all integrals in equation (21). Hence, we propose to consider equation (21) as the most general integral form of the Poynting theorem, which includes the cases of both fixed and variable with time volume of integration.

The second problem is illustrated in figure 2. It shows positions of the left and right plates of C at the time moments t and $t + \Delta t$, correspondingly, for $\Delta t < h/v$, where h is the width of C. During this short time interval, the EM energy density decreases from u to zero in the dark grey (left) region, and increases from zero to the same value u in the light grey (right) region. Thus, the EM energy density simultaneously varies in two spatially separated regions without any EM energy flux between them ($\boldsymbol{S} = 0$). Equations (21)–(26) indicate that there is no problem to describe such a variation of EM energy density in a consistent way. However,

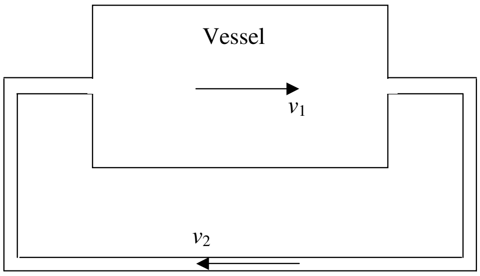

Figure 3. Circulation of water in a closed tube with a variable cross-section. For incompressible liquid (ρ = const.), its flow velocity is inversely proportional to the area of cross-section of the tube and $v_1 \ll v_2$.

in the authors' opinion, this result is somewhat at odds with a spirit of causal treatment of classical fields.

3.2. The case where vectors $\boldsymbol{E}$ and $\boldsymbol{v}$ constitute the angle $0 < \alpha < \pi/2$ (figure 1(b))

In this case, the Poynting vector $\boldsymbol{S}$ does not vanish inside the capacitor, and in the non-relativistic limit ($\gamma = \sqrt{1 - v^2/c^2} \approx 1$) it is collinear to the capacitor plates. In spite of the inequality $\boldsymbol{S} \neq 0$, one can show in a similar way, like in subsection 3.1, that the EM field inside C changes its spatial location solely due to the work done on its plates. Nevertheless, the presence of EM flux density inside C makes the problem in figure 1(b) to be qualitatively different from figure 1(a), in particular, due to the non-vanished component of $\boldsymbol{S}$ onto the y-axis. When the ratio of transverse and longitudinal sizes of the capacitor is very small, almost the entire EM energy is concentrated in its inner volume. One may assume that the same is true for the EM momentum density $\boldsymbol{p}_{\mathrm{EM}} = \boldsymbol{S}/c^2$. Thus it seems, that the entire EM momentum $\boldsymbol{P}_{\mathrm{EM}} = \int_{\mathrm{entirespace}} \boldsymbol{p}_{\mathrm{EM}}\,\mathrm{d}V$ also has the non-vanished y-component. We have shown in [3] that $(P_{\mathrm{EM}})_y$ determines the y-component of the total momentum of configuration $\boldsymbol{P}_t$. On the other hand, the Lorentz transformation for the total energy E_t and total momentum $\boldsymbol{P}_t$ between the rest frame of C and a laboratory frame gives the equality $(P_t)_y = 0$. Thus we certainly obtain a paradoxical result. How should we resolve this paradox?

Again we emphasize that almost the entire EM energy is convicted between the plates. Hence, we tacitly concluded in [3] that the same is true for the EM momentum. However, such a conclusion is, in general, erroneous. To show this, let us use an analogy from fluid mechanics and consider circulation of water in a closed tube to be shown in figure 3. Let us associate the density of water ρ with the EM energy density u and momentum density of water $\rho\boldsymbol{v}$ ($\boldsymbol{v}$ being its flow velocity) with $\boldsymbol{S}/c^2$. Further, we assume that the area of cross-section of the vessel in figure 3 is much larger than the area of the cross-section of other segments of the closed tube. Then we can certainly assert that the mass of the water inside the vessel exceeds many times the mass of the water, circulated in other segments of the tube. (There is a full analogy with the previously obtained result for a parallel-plate capacitor: almost the entire EM energy is located between its plates). However, it does not mean as yet that the total momentum of water in the main is determined by the vessel part. For the circulating flow of water, its total momentum is equal to zero and hence, the momentum of water inside the vessel is equal in value and opposite in sign to the resultant momentum of water in other parts of the closed tube. By analogy, the same conclusion can be made on the EM momentum of the capacitor: the EM energy flux, passing through its inner volume, 'scatters' into the entire space due to the leakage electric and magnetic fields, so that the total flow of EM energy has a circulative character. Hence, the entire momentum of EM field of the moving charged capacitor in figure 1(b) should be vanished, including its y-component: $(P_{\mathrm{EM}})_y = 0$. Hence $(P_t)_y = 0$, too. By such a way we resolve the paradox, but this does not disclose physical features of this problem as yet. In particular, one can argue that for the closed tube of figure 3, a flow velocity of incompressible water in the vessel is much less than the velocity of water in another tube segment due to Bernoulli's law. In contrast, one can see that *the effective velocity* of EM energy flow $\boldsymbol{V}$ (equation (20) is not substantially different inside and outside the capacitor: a drastic decrease of the value of $\boldsymbol{S}$ outside the capacitor in comparison with its inner part is accompanied by a corresponding drastic decrease of u, so that the ratio $\boldsymbol{V} = \boldsymbol{S}/u$ is not changed significantly in the entire space. However, this observation simply means that further analogy between the flow of water and flow of EM energy is no longer correct. Namely, considering a flow of water, we implied that its density $\rho = \text{const.}$ along the tube and $v_1 \ll v_2$. However, the same requirement $u = \text{const.}$ obviously is not fulfilled for the EM field of a moving capacitor: the energy density inside and outside the capacitor can vary within many orders of magnitude. Thus, Bernoulli's law (or any analogy) is not applicable to a flow of EM energy. It means that we cannot imagine a circulation of EM energy similar to the circulation of something, such as liquid, and thus, in many cases our intuition is useless to understand the processes of flow of EM energy in space. This circumstance allowed us to formulate the paradox just considered. This paradox even does not appear, if we directly integrate the Poynting vector of a parallel-plate charged capacitor over the entire space. This problem is rather technical, than physical, and we omit such calculations, presenting only a final result: we indeed obtain that $(P_{\mathrm{EM}})_y = \int_{\mathrm{entirespace}} (S_y/c^2)\,\mathrm{d}V = 0$ (taking a small, but final width of each capacitor's plate). Nevertheless, the 4-vector transformation is not fulfilled for E_t and $\boldsymbol{P}_t$ due to some extra contribution of the x-component (the axis of motion) of $\boldsymbol{P}_{\mathrm{EM}}$ into $\boldsymbol{P}_t$. This result is not surprising and has the same origin, like the familiar '4/3' puzzling for the classical electron. Physically it signifies that the quantities designated by us E_t and $\boldsymbol{P}_t$ are not, in fact, the total energy and total momentum of configuration, respectively: in classical field theory, they do not include the energy–momentum of 'Poincaré stresses' [5]. This problem is discussed over a century (see, e.g. [10]), and its further analysis falls outside the scope of the present paper.

Comparing again figures 1(a) and (b), we see that two different spatial orientations of a capacitor ($\alpha = 0$, $0 < \alpha < \pi/2$) give, in fact, two different physical problems, where different apparent paradoxes have been formulated. It reflects a known fact mentioned in [3] that spatial rotations do modify the EM energy–momentum tensor $T^{\mu\nu}_{\mathrm{EM}}$. We mentioned above that the matter tensor $T^{\mu\nu}_{\mathrm{M}}$ of gas remains unchanged under such spatial rotations. However, as we have seen, such a non-equivalence of $T^{\mu\nu}_{\mathrm{EM}}$ and $T^{\mu\nu}_{\mathrm{M}}$ does not create any additional difficulties in the analysis of both problems in subsections 3.1 and 3.2.

Having considered the apparent paradoxes, dealing with the EM energy flux, below we take the opportunity to make some general remarks on the subject and to attract the reader's attention to an inconsistency in the standard derivation of the EM energy–momentum tensor.

4. Notes on electromagnetic energy flux and the energy–momentum tensor

The Poynting theorem involves the divergence of the EM energy flux density and hence it keeps its form, if an arbitrary divergenceless vector is added to $\boldsymbol{S}$. Thus, the energy flux density $\boldsymbol{S}$ is not uniquely defined but admits a replacement $\boldsymbol{S}' = \boldsymbol{S} + \boldsymbol{C}$ with $\nabla \cdot \boldsymbol{C} = 0$. In particular, the authors of references [11–17] suggested various new expressions for $\boldsymbol{S}'$, which disagree in the direction of energy flow [18]. At the same time, one should remember that the EM energy density and energy flux density are directly determined by the time-like components of EM energy–momentum tensor $T_{\mathrm{EM}}^{\mu 0}$. Therefore, any modification of $\boldsymbol{S}$ implies a corresponding change of $T_{\mathrm{EM}}^{\mu \nu}$. However, the authors of [11–17] did not offer any physical reasons for correction of the available EM energy–momentum tensor. The latter is defined through a postulated Lagrangian density for an EM field and thus, any possible modification of $T_{\mathrm{EM}}^{\mu \nu}$ inevitably touches the foundations of classical electrodynamics. There are two methods for derivation of an energy–momentum tensor from the postulated Lagrangian density: minimization of action (classical method), as well as Hilbert's method, which uses the invariance of action under a small variation of spacetime coordinates (see, e.g. [19]). The former method, when applied to the EM field, gives a canonical energy–momentum tensor

$$T_{\mathrm{EM}}^{\mu \nu}=-\frac{1}{4\pi}\partial^{\mu}A^{\gamma}F_{\gamma}^{\nu}+\frac{1}{16\pi}g^{\mu\nu}F_{\gamma\alpha}F^{\gamma\alpha}, \tag{28}$$

which is not symmetric. (Here $F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}$ is the tensor of the EM field, A^{μ} is the four-potential, and $g^{\mu\nu}$ is the metric tensor, and $\partial^{\mu}=\partial/\partial x_{\mu}$). The Hilbert method allows a direct derivation of the symmetric EM energy–momentum tensor. A requirement of symmetry of an energy–momentum tensor results from the angular momentum conservation law, and thus, further transformation of the canonical energy–momentum tensor (28) to a symmetric form is required[4]. For this purpose, a tensorial gauge transformation should be applied

$$T_{\mathrm{EM}}^{\mu\nu}\rightarrow T_{\mathrm{EM}}^{\mu\nu}+\partial_{\gamma}\psi^{\mu\nu\gamma}\quad(\text{where } \psi^{\mu\nu\gamma}=-\psi^{\mu\gamma\nu}). \tag{29}$$

Below we indicate an inconsistency, which exists to the moment in the adopted derivation of the symmetric EM energy momentum tensor on the basis of equations (28) and (29). The Hilbert method will be discussed elsewhere.

So, we need to modify the tensor (28) to a symmetric form for an arbitrary system of charged particles. Choosing the gauge function as

$$\psi_{\gamma}^{\mu\nu}=\frac{1}{4\pi}A^{\mu}F_{\gamma}^{\nu} \tag{30}$$

and writing

$$\partial^{\gamma}\psi_{\gamma}^{\mu\nu}=\frac{1}{4\pi}(\partial^{\gamma}A^{\mu})F_{\gamma}^{\nu}+\frac{1}{4\pi}A^{\mu}\left(\partial^{\gamma}F_{\gamma}^{\nu}\right), \tag{31}$$

we can transform the tensor (28) to the well-known symmetric form

$$T_{\mathrm{EM}}^{\mu\nu}=\frac{1}{4\pi}\left(-F^{\mu\gamma}F_{\gamma}^{\nu}+\frac{1}{4}g^{\mu\nu}F_{\gamma\alpha}F^{\gamma\alpha}\right), \tag{32}$$

if we apply *the homogeneous* Maxwell's equation

$$\partial_{\gamma}F^{\nu\gamma}=0. \tag{33}$$

Equations (28)–(33) are presented in many textbooks and papers, and it can be wondered, why an obvious inconsistency in this derivation has not been commented on until now: the homogeneous Maxwell equation (33) has been applied to the system of charged particles. This inconsistency becomes especially clearly seen, when one derives the motional equation of charged particles with the tensor (32): the reader can check that application of a relevant *inhomogeneous* Maxwell equation

$$\partial_{\gamma}F^{\gamma\nu}=\frac{4\pi}{c}j^{\nu} \tag{34}$$

(j^{ν} being the four-current density) is essential to obtain a four-dimensional Lorentz force law in its conventional form (see, e.g. [6, 10, 19]). Thus, the same equation (34) should be certainly applied in the tensorial gauge transformation instead of (33). We invite the reader to explore this problem.

[4] The authors of [20] advanced and substantiated an interesting method, where the canonical EM energy momentum tensor can be applied for a classical description of the electron's spin. However, as is usual for classical field theory, now we consider the charged spinless particles.

5. Concluding remarks

With this paper, we finalize a series of publications on apparent paradoxes in classical electrodynamics and their resolution. We hope that this series was helpful to elucidate some subtle problems of classical physics, which can be accompanied by various paradoxical situations. Simultaneously, we emphasize that the paradoxes formulated are actually *apparent*. All of them disappear, when a consistent approach of relativistic physics is applied. We believe that the presented resolutions of these paradoxes once more demonstrate that modern classical electrodynamics is a powerful theory within its domain of applicability. At the same time, one should recognize that nowadays this theory is not consistent on the whole, and there exist some *actual* unresolved problems.

(1) The infinite electromagnetic self-energy and electromagnetic mass of a classical charged particle, when its radius tends to zero (the latter is required by the special relativity theory).
(2) The infinite electromagnetic self-force, exerting on a point-like charge by its own bound EM field.
(3) The presence of the so-called 'runaway solutions': e.g., a self-acceleration of charged particle by the force of reaction of its radiation.

A procedure of standard renormalization (see, e.g., [4, 10]) allows eliminating the infinite electromagnetic mass of charged particles, but other mentioned difficulties remain in force. Such a situation is usually explained by the well-known fact that the classical theory cannot be applied to very small spatial domains, giving way to the quantum electrodynamics. This statement is undoubtedly true, but nevertheless, we express a hope that the elimination of the revealed inconsistency in derivation of the EM energy–momentum tensor (where equation (33) should be replaced by equation (34), perhaps, would be helpful for further progress in the analysis of the mentioned problems 1–3.

Another crucial question can be formulated in the following way: does classical electrodynamics always adequately describe the phenomena of electricity and magnetism in a macro-scale domain, and what is an actual boundary between classical and quantum electrodynamics? The very recent experiments [21, 22] indicate non-local properties of the bound electromagnetic field, which is obviously at odds with a classical causality. Even so, we have to distinguish the problems of *intrinsic logical structure* and *domain of applicability* for each theory. The series of our papers on apparent paradoxes in classical electrodynamics is relevant only to the former problem and gives an additional demonstration, regarding how the logic of classical field theory *is applied* to particular physical problems.

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